This week

1. Section 12.1: coordinate systems
2. Section 12.2: vectors
Choose an origin.

Choose an $x$-axis. The arrow head indicates the positive part.

The $y$-axis is now fixed:
- The $y$-axis is perpendicular on the $x$-axis.
- The positive $y$-axis is obtained from the positive $x$-axis by rotating it $90^\circ$ to the left.

Let $P$ be a point in the plane.

Find the projections $p_1$ and $p_2$ of $P$ on the $x$-axis and $y$-axis.

The Cartesian coordinates of $P$ are $p_1$ and $p_2$, notation:
$$ P = (p_1, p_2). $$

Every $P \in \mathbb{R}^2$ corresponds to a unique pair of Cartesian coordinates.
Distance in $\mathbb{R}^2$

![Distance in $\mathbb{R}^2$](image)

**Definition**

The distance between two points $P_1$ and $P_2$ with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
The $xy$-plane is the plane that contains the $x$-axis and the $y$-axis.

The $yz$-plane is the plane that contains the $y$-axis and the $z$-axis.

The $xz$-plane is the plane that contains the $x$-axis and the $z$-axis.

In $\mathbb{R}^3$ points have three coordinates:

\[ P = (p_1, p_2, p_3). \]

The projection of $P$ on the $xy$-plane is

\[ P' = (p_1, p_2, 0). \]

Regard $P$ as the vertex of a rectangular block.
Distances in $\mathbb{R}^3$

Definition

The distance between two points $P_1$ and $P_2$ with coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 1.8

Find the distance between $P_1 = (2, 1, 5)$ and $P_2 = (-2, 3, 0)$.

- $|P_1P_2| = \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2}$
  
  $= \sqrt{16 + 4 + 25}$
  
  $= \sqrt{45} = 3\sqrt{5}$.

- Different style:
  
  $|P_1P_2|^2 = (-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2$
  
  $= 16 + 4 + 25$
  
  $= 45 = 3^2 \cdot 5$,

  hence $|P_1P_2| = 3\sqrt{5}$.
Exercises

1 In $\mathbb{R}^2$, find the distance between the given points.
   - $P_1 = (-1, 3)$ and $P_2 = (2, 6)$.
   - $P_1 = \left(\frac{3}{2}, -\frac{5}{2}\right)$ and $P_2 = \left(-\frac{3}{2}, \frac{3}{2}\right)$.

Assignment: IMM1 - Tutorial 4.1

Vectors

- A vector is a geometric object with a magnitude and a direction.
- A vector can be regarded as a directed line segment, and can be drawn as an arrow.
- Every vector has an initial- and a terminal point.
- The vector with initial point $P$ and terminal point $Q$ is denoted as $\overrightarrow{PQ}$.
- The name of a vector is bold in printed text, or underlined in handwritten texts: $\mathbf{v}$. 
Length 2.2

Definition

The length of the vector \( \overrightarrow{PQ} \) is defined as the distance between \( P \) and \( Q \).

- If \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) then the length of \( \mathbf{v} = \overrightarrow{PQ} \) is
  \[
  |\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
  \]
- If \( P = (x_1, y_1, z_1) \) and \( Q = (x_2, y_2, z_2) \) then the length of \( \mathbf{v} = \overrightarrow{PQ} \) is
  \[
  |\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
  \]

Example 2.2. Section 12.2, example 1(b)

Find the length of the vector \( \overrightarrow{PQ} \) where \( P = (-3, 4, 1) \) and \( Q = (-5, 2, 2) \).

\[
|\overrightarrow{PQ}| = \sqrt{(-5 - (-3))^2 + (2 - 4)^2 + (2 - 1)^2}
= \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = 3.
\]

Equality of vectors 2.3

Definition

Two vectors are equal if they have the same length and the same direction.

Theorem

Two vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{P'Q'} \) are equal if \( PQQ'P' \) are the vertices of a parallelogram.

- The quadrilateral \( OAQP \) is a parallelogram.
Component form of a vector

**Definition**

*Een vector is said to be in standard position if the origin is the initial point of the vector.*

**Theorem**

*For every vector \( \mathbf{v} \) there is exactly one point \( A \) such that \( \mathbf{v} = \overrightarrow{OA} \).*

**Definition**

*The component form of a vector \( \mathbf{v} = \overrightarrow{OA} \) is the sequence \( \langle v_1, \ldots, v_n \rangle \), where \( v_1, \ldots, v_n \) are the coordinates of \( A \).*

\[
\overrightarrow{PQ} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle.
\]

**Proof for \( n = 2 \):**

- Put \( \overrightarrow{PQ} \) in standard position: \( \overrightarrow{PQ} = \overrightarrow{OV} = \langle v_1, v_2 \rangle \) met \( \mathbf{v} = \langle v_1, v_2 \rangle \).
- Then \( q_1 = p_1 + v_1 \), in other words: \( v_1 = q_1 - p_1 \).
- This holds for all coordinates: \( v_k = q_k - p_k \) \( (k = 1, 2) \).
- The proof for \( n = 3 \) is analogous.
Theorem

Let \( \langle u_1, \ldots, u_n \rangle \) be the component form of a vector \( \mathbf{u} \). Then the length of \( \mathbf{u} \) is

\[
|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.
\]

Proof:

- Put \( \mathbf{u} \) in standard position: \( \mathbf{u} = \overrightarrow{OP} \) where \( P \) is a point with coordinates \( (u_1, \ldots, u_n) \).
- Then

\[
|\mathbf{u}| = |\overrightarrow{OP}| = \sqrt{(u_1 - 0)^2 + \cdots + (u_n - 0)^2} = \sqrt{u_1^2 + \cdots + u_n^2}.
\]

Unit vectors

Definition

A vector \( \mathbf{u} \) is called a **unit vector** if the length of \( \mathbf{u} \) is equal to 1.

Example

- Is \( \mathbf{u} = \langle 2, -1, -2 \rangle \) a unit vector?
- Is \( \mathbf{v} = \langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle \) a unit vector?

\[
|\mathbf{u}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3 \neq 1,
\]

hence \( \mathbf{u} \) is not a unit vector.

\[
|\mathbf{v}| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = 1,
\]

hence \( \mathbf{v} \) is a unit vector.
Standard unit vectors

**Definition**

- In $\mathbb{R}^2$ we define the **standard unit vectors** $\mathbf{i}$ and $\mathbf{j}$ by 
  \[ \mathbf{i} = \langle 1, 0 \rangle \text{ and } \mathbf{j} = \langle 0, 1 \rangle. \]
- In $\mathbb{R}^3$ we define the **standard unit vectors** $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ by 
  \[ \mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle \text{ and } \mathbf{k} = \langle 0, 0, 1 \rangle. \]

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Standard unit vectors and component form

**Notation**

- In $\mathbb{R}^2$, every vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is written as 
  \[ \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}. \]
- In $\mathbb{R}^3$, every vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is written as 
  \[ \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \]

- Sloppy notation:
  \[ \langle 1, 0, -1 \rangle = 1 \mathbf{i} + 0 \mathbf{j} + (-1) \mathbf{k} = \mathbf{i} - \mathbf{k}. \]
  - The sloppy notation is justified by the fact that you can scale and add vectors (next lecture).
Exercises

Assignment: IMM1 - Tutorial 4.2

* in exercises 25, 27, and 29 you only need to calculate the length of the given vectors.

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The zero vector

**Definition**

The zero vector, denoted as \( \mathbf{0} \), is the vector \( \overrightarrow{PP} \), where \( P \) is an arbitrary point.

- The zero vector has no direction and is therefore denoted as a point.
- The component form of the zero vector is \( \overrightarrow{OO} = \langle 0, \ldots, 0 \rangle \).
- The zero vector has length zero: \( |\mathbf{0}| = \sqrt{0^2 + \cdots + 0^2} = 0 \).
- The zero vector is the only vector with length 0.
The inverse of a vector

**Definition**

The inverse of \( \mathbf{v} = \overrightarrow{PQ} \) is the vector \( -\mathbf{v} = \overrightarrow{QP} \).

- If \( \mathbf{v} = (v_1, \ldots, v_n) \), then \( -\mathbf{v} = (-v_1, \ldots, -v_n) \).
- The vectors \( \mathbf{v} \) and \( -\mathbf{v} \) have the same length:
  \[
  |-\mathbf{v}| = \sqrt{(-v_1)^2 + \cdots + (-v_n)^2} = \sqrt{v_1^2 + \cdots + v_n^2} = |\mathbf{v}|.
  \]

Scalar multiplication

**Definition**

Let \( \mathbf{v} \) be a vector and let \( \alpha \) be a real number. The scalar product \( \alpha \mathbf{v} \) is defined as follows:

- If \( \alpha > 0 \), then \( \alpha \mathbf{v} \) has the same direction as \( \mathbf{v} \), but it is \( \alpha \) times as long as \( \mathbf{v} \).
- If \( \alpha = 0 \), then \( \alpha \mathbf{v} = \mathbf{0} \).
- If \( \alpha < 0 \) then \( \alpha \mathbf{v} \) the inverse of \( |\alpha| \mathbf{v} \), hence \( \alpha \mathbf{v} = -|\alpha| \mathbf{v} \).

**Theorem**

If \( \mathbf{v} = (v_1, \ldots, v_n) \), then \( \alpha \mathbf{v} = (\alpha v_1, \ldots, \alpha v_n) \) for all real numbers \( \alpha \).

- The number \( \alpha \) is called a scalar.
Example (in $\mathbb{R}^2$)

Let $\mathbf{v} = \overrightarrow{PQ}$ with $P = (3, 2)$ and $Q = (1, -2)$. Find the terminal point of the vector $-\frac{1}{2} \mathbf{v}$ with initial point $P$.

- The component form of $\mathbf{v}$ is $\langle 1 - 3, -2 - 2 \rangle = \langle -2, -4 \rangle$.
- The component form of $-\frac{1}{2} \mathbf{v}$ is $\langle \left(-\frac{1}{2}\right) \cdot (-2), \left(-\frac{1}{2}\right) \cdot (-4) \rangle = (1, 2)$.
- Let $-\frac{1}{2} \mathbf{v} = \overrightarrow{PX}$ with $X = (x, y)$, then $\langle 1, 2 \rangle = \langle x - 3, y - 2 \rangle$, hence $x = 4$ and $y = 4$ and therefore $X = (4, 4)$.

Unit vectors

Recap

A vector $\mathbf{u}$ is called a unit vector if the length of $\mathbf{u}$ is equal to 1.

Theorem

If $\mathbf{u} \neq 0$, then $\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|} \mathbf{u}$ is a unit vector.

- We sometimes use the sloppy notation $\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$.

Definition

The vector $\hat{\mathbf{u}}$ is called the direction of $\mathbf{u}$.
Example

Let \( P_1 = (1, 0, 1) \) and \( P_2 = (3, 2, 0) \). Find the direction from \( P_1 \) to \( P_2 \).

- Define \( \mathbf{u} = P_1P_2 \):
  \[
  \mathbf{u} = (3 - 1, 2 - 0, 0 - 1) = (2, 2, -1).
  \]
- The vector \( \mathbf{u} \) is not a unit vector:
  \[
  |\mathbf{u}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = 3.
  \]
- Scale \( \mathbf{u} \) with \( 1/|\mathbf{u}| \):
  \[
  \hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|} \mathbf{u} = \frac{1}{3} \mathbf{u} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle.
  \]
- The direction of from \( P_1 \) to \( P_2 \) is the unit vector
  \[
  \hat{\mathbf{u}} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle.
  \]

Addition

Definition – head-to-tail construction

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors. The sum of \( \mathbf{u} \) and \( \mathbf{v} \) (notation: \( \mathbf{u} + \mathbf{v} \)) is defined as follows:

1. Write \( \mathbf{u} = \overrightarrow{PQ} \) for two points \( P \) and \( Q \).
2. Find a point \( R \) such that \( \mathbf{v} = \overrightarrow{QR} \).
3. Define \( \mathbf{u} + \mathbf{v} = \overrightarrow{PR} \).
Theorem – component-wise addition

Let \( u = \langle u_1, \ldots, u_n \rangle \) and \( v = \langle v_1, \ldots, v_n \rangle \), then the component form of the sum of \( u \) and \( v \) is

\[
u + v = \langle u_1 + v_1, \ldots, u_n + v_n \rangle.
\]

Proof:

- Let \( u = \overrightarrow{PQ} \) and \( v = \overrightarrow{QR} \) where \( P = (p_1, \ldots, p_n) \), \( Q = (q_1, \ldots, q_n) \) and \( R = (r_1, \ldots, r_n) \), then
  \[
u_i = q_i - p_i \quad \text{and} \quad v_i = r_i - q_i \quad (i = 1, \ldots, n).
\]
- Head-to-tail addition gives \( u + v = \overrightarrow{PR} = \langle x_1, \ldots, x_n \rangle \) with \( x_i = r_i - p_i \).
- \[
u_i + v_i = (q_i - p_i) + (r_i - q_i) = r_i - p_i = x_i \quad \text{for all } i = 1, \ldots, n.
\]

Theorem – parallelogram law

Let \( u = \overrightarrow{OP} \) and \( v = \overrightarrow{OQ} \) be two vectors in standard position. Let \( R \) be the terminal point of the component form of \( u + v \), then \( O, P, Q \) and \( R \) are the vertices of a parallelogram.

- Since \( v = \overrightarrow{PR} \), the line segment \( OQ \) is parallel to \( PR \), and \( OQ \) and \( PR \) have the same length.

Corollary – commutativity of the addition

For all vectors \( u \) and \( v \) we have \( u + v = v + u \).

- Vector \( \overrightarrow{QR} \) has the same length and direction as \( \overrightarrow{OP} = u \), hence \( u + v = \overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = v + u \).
Properties

Let $u$, $v$ and $w$ be vectors, and let $a$ and $b$ be scalars, then

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = u$
4. $u + (-u) = 0$
5. $0u = 0$
6. $1u = u$
7. $a(bu) = (ab)u$
8. $a(u + v) = au + av$
9. $(a + b)u = au + bu$

Property 2 is called the **associativity** of the addition, and it justifies the notation $u + v + w$.

Let $u$ and $v$ be vectors, and let $a$ be a scalar, then

1. $|0| = 0$
2. If $|u| = 0$ then $u = 0$
3. $|au| = |a||u|$
4. $|u + v| \leq |u| + |v|$

Subtraction

**Definition – head-to-tail construction**

Let $u$ and $v$ be two vectors. The **difference of $u$ and $v$** (notation: $u - v$) is defined as follows:

1. Write $u = \overrightarrow{PQ}$ for two points $P$ and $Q$.
2. Find a point $R$ such that $v = \overrightarrow{PR}$.
3. Define $u - v = \overrightarrow{RQ}$.

Note that from the haid-to-tail law follows: $v + (u - v) = u$. 
Subtraction

Theorem – component-wise subtraction

Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, then

$$u - v = (u_1 - v_1, \ldots, u_n - v_n).$$

Theorem – parallelogram law, extended version

Let $u = \overrightarrow{PQ}$ and $v = \overrightarrow{PR}$ be vectors with common initial point $P$. Let $S$ be defined by assuming that $PQSR$ is a parallelogram. Then

$$u + v = \overrightarrow{PS}, \quad u - v = \overrightarrow{RQ} \quad \text{and} \quad v - u = \overrightarrow{QR}.$$

Standard unit vectors and component form

Theorem

- In $\mathbb{R}^2$, for every vector $v = (v_1, v_2)$ we have
  $$v = v_1 \mathbf{i} + v_2 \mathbf{j}.$$
- In $\mathbb{R}^3$, for every vector $v = (v_1, v_2, v_3)$ we have
  $$v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

Proof in $\mathbb{R}^2$:

$$v = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j}.$$
Exercises

1. Show that if \( \mathbf{v} = (v_1, v_2, v_3) \), then \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \).

Assignment: IMM1 - Tutorial 4.3

* in exercises 25, 27, and 29 you only need to calculate the direction of the given vectors.

Decomposition in \( \mathbb{R}^2 \)

**Definition**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors in \( \mathbb{R}^2 \).

- A linear combination of \( \mathbf{u} \) and \( \mathbf{v} \) is an expression
  \[ \alpha \mathbf{u} + \beta \mathbf{v}, \]
  where \( \alpha \) and \( \beta \) are arbitrary real numbers.

- Given vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{x} \), then a decomposition of \( \mathbf{x} \) along \( \mathbf{u} \) and \( \mathbf{v} \) is a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \) that yields \( \mathbf{x} \), in other words: find numbers \( \alpha \) and \( \beta \) such that
  \[ \mathbf{x} = \alpha \mathbf{u} + \beta \mathbf{v}. \]

- A decomposition does not always exist.
- A decomposition is not always unique.
Decomposition in $\mathbb{R}^2$

**Example**

Let $u = \langle 1, 2 \rangle$ and let $v = \langle 3, 2 \rangle$. Decompose $x = \langle 3, 4 \rangle$ along $u$ and $v$.

- Assume $x = au + bv$, then
  
  $$x = \langle 3, 4 \rangle = au + bv = a\langle 1, 2 \rangle + b\langle 3, 2 \rangle = \langle a + 3b, 2a + 2b \rangle.$$  

- This yields a **system** of equations:
  
  $$\begin{cases}
  a + 3b = 3, \\
  2a + 2b = 4.
  \end{cases}$$

- From the first equation follows $a = 3 - 3b$.  

- From the second equation we derive
  
  $$2(3 - 3b) + 2b = 4 \iff -4b = -2 \iff b = \frac{1}{2}$$

- Equation (1) gives $a = 3 - \frac{3}{2} \iff a = \frac{3}{2}$

Orthogonal decomposition in $\mathbb{R}^2$

**Definition**

Let $x$ be a vector in $\mathbb{R}^2$. The **orthogonal decomposition** of $x$ is a decomposition along $i$ and $j$.

**Theorem**

Let $x$ be a non-zero vector in $\mathbb{R}^2$. The orthogonal decomposition of $x$ is

$$x = |x| \cos \theta i + |x| \sin \theta j,$$

where $\theta$ is the angle that $x$ makes with the positive $x$-axis.
Example

A 75 N weight is suspended by two wires as shown in the picture above. Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.

Newton’s first law

If an object is at rest, the total sum of the forces exerted on the object is equal to zero.

Example (continued)

- The force caused by gravitation is $\mathbf{F}_G = \langle 0, -75 \rangle$.
- From Newton’s first law follows: $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_G = 0$, hence $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75 \rangle$.
- Decompose $\mathbf{F}_1$ and $\mathbf{F}_2$:
  \[ \mathbf{F}_1 = -|\mathbf{F}_1| \cos 55^\circ \mathbf{i} + |\mathbf{F}_1| \sin 55^\circ \mathbf{j} \]
  and
  \[ \mathbf{F}_2 = |\mathbf{F}_2| \cos 40^\circ \mathbf{i} + |\mathbf{F}_2| \sin 40^\circ \mathbf{j}. \]
Example (continued)

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In component form the equation becomes

\[
\begin{align*}
-|\mathbf{F}_1| \cos 55^\circ + |\mathbf{F}_2| \cos 40^\circ &= 0 \\
|\mathbf{F}_1| \sin 55^\circ + |\mathbf{F}_2| \sin 40^\circ &= 75
\end{align*}
\]

From the first equation follows

\[|\mathbf{F}_2| = \frac{|\mathbf{F}_1| \cos 55^\circ}{\cos 40^\circ}.\]

Example (continued)

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Substitute this in the second equation:

\[|\mathbf{F}_1| \sin 55^\circ + \frac{|\mathbf{F}_1| \cos 55^\circ}{\cos 40^\circ} \sin 40^\circ = 75\]

Solving for $|\mathbf{F}_1|$: \[|\mathbf{F}_1| = \frac{75}{\sin 55^\circ + \cos 55^\circ \tan 40^\circ} \approx 57.67 \text{ N}\]

Consequently \[|\mathbf{F}_2| = \frac{75 \cos 55^\circ}{\sin 55^\circ \cos 40^\circ + \cos 55^\circ \sin 40^\circ} \approx 43.18 \text{ N}\]
Assignment: IMM1 - Tutorial 4.4